

Completeness for Concurrent Kleene Algebra

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NII Logic Seminar

Introduction

Kleene Algebra models *program flow*.

- abort (0) and skip (1)
- atomic actions ($a, b, \dots$)
- non-deterministic choice ($+$)
- sequential composition ($\cdot$)
- indefinite repetition ($*$)

$$(e + f)^* \equiv_{\text{KA}} e^* \cdot (f \cdot e^*)^*$$

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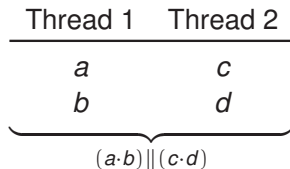
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Thread 1	Thread 2
a	c
b	d

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Concurrent KA¹ adds *parallel composition* ($\parallel$)

¹Hoare, Möller, Struth, and Wehrman 2009

Introduction

KA is well-studied:

- Decision procedures
- Coalgebra, automata
- Axiomatisation of equivalence

[Hopcroft and Karp 1971; Bonchi and Pous 2013]

[Kleene 1956; Brzozowski 1964; Silva 2010]

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CKA is a work in progress:

- Decision procedures
- Coalgebra, automata
- Axiomatisation of equivalence

[Brunet, Pous, and Struth 2017]

[K., Brunet, Luttik, Silva, and Zanasi 2017]

[Gischer 1988; Laurence and Struth 2014]

Theorem (Kozen 1994)

The axioms for KA are complete for equivalence:

$$e \equiv_{\text{KA}} f \iff \llbracket e \rrbracket_{\text{KA}} = \llbracket f \rrbracket_{\text{KA}}$$

$\llbracket - \rrbracket_{\text{KA}}$ is the regular language interpretation of e .

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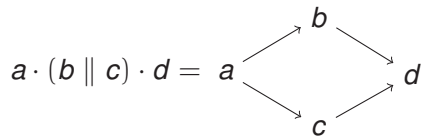
Question

Can we find axioms for CKA that are complete for equivalence? That is,

$$e \equiv_{\text{CKA}} f \stackrel{?}{\iff} \llbracket e \rrbracket_{\text{CKA}} = \llbracket f \rrbracket_{\text{CKA}}$$

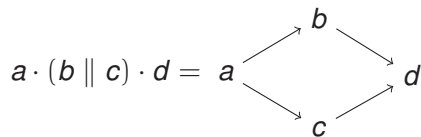
$\llbracket - \rrbracket_{\text{CKA}}$ is a generalized regular language interpretation of e .

- Pomset: “word with parallelism”



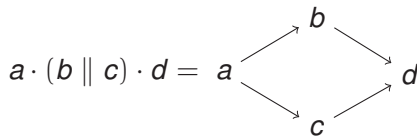
Preliminaries

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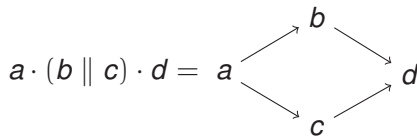
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- Composition lifts:

- $\mathcal{U} \cdot \mathcal{V} = \{U \cdot V : U \in \mathcal{U}, V \in \mathcal{V}\}$

- $\mathcal{U} \parallel \mathcal{V} = \{U \parallel V : U \in \mathcal{U}, V \in \mathcal{V}\}$

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 - $\mathcal{U} \parallel \mathcal{V} = \{U \parallel V : U \in \mathcal{U}, V \in \mathcal{V}\}$
- Kleene star: $\mathcal{U}^* = \bigcup_{n < \omega} \mathcal{U}^n$

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$\mathcal{T}$ is the set generated by the grammar

$$e, f ::= 0 \mid 1 \mid a \in \Sigma \mid e + f \mid e \cdot f \mid e \parallel f \mid e^*$$

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BKA *semantics* is given by $\llbracket - \rrbracket_{\text{BKA}} : \mathcal{T} \rightarrow 2^{\text{Pom}_{\Sigma}}$.

$$\llbracket 0 \rrbracket_{\text{BKA}} = \emptyset$$

$$\llbracket 1 \rrbracket_{\text{BKA}} = \{1\}$$

$$\llbracket a \rrbracket_{\text{BKA}} = \{a\}$$

$$\llbracket e + f \rrbracket_{\text{BKA}} = \llbracket e \rrbracket_{\text{BKA}} \cup \llbracket f \rrbracket_{\text{BKA}}$$

$$\llbracket e \cdot f \rrbracket_{\text{BKA}} = \llbracket e \rrbracket_{\text{BKA}} \cdot \llbracket f \rrbracket_{\text{BKA}}$$

$$\llbracket e \parallel f \rrbracket_{\text{BKA}} = \llbracket e \rrbracket_{\text{BKA}} \parallel \llbracket f \rrbracket_{\text{BKA}}$$

$$\llbracket e^* \rrbracket_{\text{BKA}} = \llbracket e \rrbracket_{\text{BKA}}^*$$

Preliminaries

Axioms for BKA :

$$e + 0 \equiv_{\text{BKA}} e$$

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$$e + f \equiv_{\text{BKA}} f + e$$

$$e + (f + g) \equiv_{\text{BKA}} (f + g) + e$$

$$e \cdot (f \cdot g) \equiv_{\text{BKA}} (e \cdot f) \cdot g$$

$$e \cdot (f + g) \equiv_{\text{BKA}} e \cdot f + e \cdot g$$

$$(e + f) \cdot g \equiv_{\text{BKA}} e \cdot g + f \cdot g$$

$$1 + e \cdot e^* \equiv_{\text{BKA}} e^*$$

$$e \cdot f + g \leq_{\text{BKA}} f \implies e^* \cdot g \leq_{\text{BKA}} f$$

$$e \parallel f \equiv_{\text{BKA}} f \parallel e$$

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$U \sqsubseteq V$: U is “more sequential” than V

- Closure* under pomset subsumption: $\mathcal{U} \downarrow = \{U' \sqsubseteq U : U \in \mathcal{U}\}$

$\mathcal{U} \downarrow$: all “sequentialisations” of pomsets in $\mathcal{U}$.

- CKA semantics: $\llbracket e \rrbracket_{\text{CKA}} = \llbracket e \rrbracket_{\text{BKA}} \downarrow$.

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- Axioms to build $\equiv_{\text{CKA}}$: all axioms for $\equiv_{\text{BKA}}$, as well as the *exchange law*:

$$(e \parallel f) \cdot (g \parallel h) \leq_{\text{CKA}} (e \cdot g) \parallel (f \cdot h)$$

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Lemma (Hoare, Möller, Struth, and Wehrman 2009)

The axioms of CKA are sound for equivalence, i.e.,

$$e \equiv_{\text{CKA}} f \implies \llbracket e \rrbracket_{\text{CKA}} = \llbracket f \rrbracket_{\text{CKA}}$$

Theorem (Kozen 1994)

Let M be an n -by- n matrix over $\mathcal{T}$, and $\vec{b}$ an n -dimensional vector over $\mathcal{T}$.

The inequation $M \cdot \vec{x} + \vec{b} \leq_{\text{KA}} \vec{x}$ admits a unique least solution (with respect to $\leq_{\text{KA}}$).

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- In fact, the solution is the same in both systems!
- We use this as a device to find specific terms later on.

Closure

Definition

Let $e \in \mathcal{T}$; a *closure* of e is a term $e\downarrow$ such that

$$1 \quad e\downarrow \equiv_{\text{CKA}} e$$

$$2 \quad \llbracket e \rrbracket_{\text{CKA}} = \llbracket e\downarrow \rrbracket_{\text{BKA}}$$

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Lemma (Laurence & Struth)

If every term e has a closure $e\downarrow$, then $\llbracket e \rrbracket_{\text{CKA}} = \llbracket f \rrbracket_{\text{CKA}}$ implies $e \equiv_{\text{CKA}} f$.

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Proof.

Observe that $\llbracket e\downarrow \rrbracket_{\text{BKA}} = \llbracket f\downarrow \rrbracket_{\text{BKA}}$, and therefore $e \equiv_{\text{CKA}} e\downarrow \equiv_{\text{BKA}} f\downarrow \equiv_{\text{CKA}} f$. □

Lemma

If e, f have closures $e\downarrow$ and $f\downarrow$ respectively, then

- 1 $e\downarrow + f\downarrow$ is a closure of $e + f$
- 2 $e\downarrow \cdot f\downarrow$ is a closure of $e \cdot f$
- 3 $e\downarrow^*$ is a closure of e^*

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One case remains: parallel composition.

Induction hypothesis: for $e \in \mathcal{T}$, we assume that:

- If f is a strict subterm of e , we can construct $f\downarrow$.
- If $|f| < |e|$ we can construct $f\downarrow$.²

² $|e|$ is the nesting level e w.r.t. $\parallel$

A *preclosure* is almost a closure, but not quite.

Definition

Let $e \in \mathcal{T}$. A *preclosure* of e is a term $\tilde{e} \in \mathcal{T}$ such that

- 1 $\tilde{e} \equiv_{\text{CKA}} e$.
- 2 if $U \in \llbracket e \rrbracket_{\text{CKA}}$ is non-sequential, then $U \in \llbracket \tilde{e} \rrbracket_{\text{BKA}}$

Definition

Let $e \in \mathcal{T}$; Δ_e is the smallest relation on $\mathcal{T}$ such that

$$\begin{array}{c}
 \frac{}{1 \Delta_e e} \qquad \frac{}{e \Delta_e 1} \qquad \frac{\ell \Delta_{e_0} r}{\ell \Delta_{e_1+e_0} r} \qquad \frac{\ell \Delta_{e_1} r}{\ell \Delta_{e_0+e_1} r} \qquad \frac{\ell \Delta_e r}{\ell \Delta_{e^*} r} \\
 \\
 \frac{\ell \Delta_{e_0} r \quad 1 \in \llbracket e_1 \rrbracket_{\text{CKA}}}{\ell \Delta_{e_0 \cdot e_1} r} \qquad \frac{\ell \Delta_{e_1} r \quad 1 \in \llbracket e_0 \rrbracket_{\text{CKA}}}{\ell \Delta_{e_0 \cdot e_1} r} \qquad \frac{\ell_0 \Delta_{e_0} r_0 \quad \ell_1 \Delta_{e_1} r_1}{\ell_0 \parallel \ell_1 \Delta_{e_0 \parallel e_1} r_0 \parallel r_1}
 \end{array}$$

Lemma

Let $V, W \neq 1$, $e \in \mathcal{T}$, and $V \parallel W \in \llbracket e \rrbracket_{\text{BKA}}$; there exist $\ell \Delta_e r$ with $V \in \llbracket \ell \rrbracket_{\text{BKA}}$ and $W \in \llbracket r \rrbracket_{\text{BKA}}$.

Definition

Let $e, f \in \mathcal{T}$; the term $e \odot f$ is defined as follows:

$$e \odot f \triangleq e \parallel f + \sum_{\substack{\ell \Delta_{e \parallel f} r \\ |\ell|, |r| < |e \parallel f|}} \ell \downarrow \parallel r \downarrow$$

Lemma

Let $e, f \in \mathcal{T}$; then

- 1 $e \odot f \equiv_{\text{CKA}} e \parallel f$
- 2 if $U \in \llbracket e \parallel f \rrbracket_{\text{CKA}}$ is non-sequential, then $U \in \llbracket e \odot f \rrbracket_{\text{BKA}}$

That is, $e \odot f$ is a preclosure of $e \parallel f$.

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Sketch: given $e \parallel f$, apply exchange law syntactically, “in the limit”.

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For instance: if $e = a \cdot b$ and $f = c \cdot d$:

$$\blacksquare (a \parallel c) \cdot (b \parallel d) \leq_{\text{CKA}} e \parallel f$$

$$(e = a \bullet b, f = c \bullet d)$$

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$$\blacksquare \dots$$

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$$\blacksquare a \cdot (b \parallel (c \cdot d)) \leq_{\text{CKA}} e \parallel f$$

$$(e = a \bullet b, f = 1 \bullet c \cdot d)$$

$$\blacksquare c \cdot ((a \cdot b) \parallel d) \leq_{\text{CKA}} e \parallel f$$

$$(e = 1 \bullet a \cdot b, f = c \bullet d)$$

$$\blacksquare \dots$$

Goal: find enough of these terms to cover all pomsets in $\llbracket e \parallel f \rrbracket_{\text{CKA}}$.

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fixpoints of inequations

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$$\begin{array}{c} \frac{}{1 \nabla_1 1} \quad \frac{}{a \nabla_a 1} \quad \frac{}{1 \nabla_a a} \quad \frac{}{1 \nabla_{e^*} 1} \quad \frac{\ell \nabla_e r}{\ell \nabla_{e+f} r} \quad \frac{\ell \nabla_f r}{\ell \nabla_{e+f} r} \\[10pt] \frac{\ell \nabla_e r}{\ell \nabla_{e \cdot f} r \cdot f} \quad \frac{\ell \nabla_f r}{e \cdot \ell \nabla_{e \cdot f} r} \quad \frac{\ell_0 \nabla_e r_0 \quad \ell_1 \nabla_f r_1}{\ell_0 \parallel \ell_1 \nabla_{e \parallel f} r_0 \parallel r_1} \quad \frac{\ell \nabla_e r}{e^* \cdot \ell \nabla_{e^*} r \cdot e^*} \end{array}$$

Lemma

Let $e \in \mathcal{T}$ and $U \cdot V \in \llbracket e \rrbracket_{\text{WCKA}}$; there exist $\ell \nabla_e r$ such that $U \in \llbracket \ell \rrbracket_{\text{CKA}}$ and $V \in \llbracket r \rrbracket_{\text{CKA}}$.

Closure

Suppose that for all $g, h \in \mathcal{T}$, we have that $X_{g \parallel h}$ is a closure of $g \parallel h$.

Then we find

$$e \parallel f + \sum_{\substack{\ell_e \nabla_e r_e \\ \ell_f \nabla_f r_f}} (\ell_e \parallel \ell_f) \cdot (r_e \parallel r_f) \leq_{\text{CKA}} X_{e \parallel f}$$

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Closure

Suppose that for all $g, h \in \mathcal{T}$, we have that $X_{g||h}$ is a closure of $g || h$.

Then we find

$$e || f + \sum_{\substack{\ell_e \nabla_e r_e \\ \ell_f \nabla_f r_f}} (\ell_e \odot \ell_f) \cdot X_{r_e || r_f} \leq_{\text{CKA}} X_{e || f}$$

For $X_{r_e || r_f}$, we find another inequation, et cetera. . .

Closure

Suppose that for all $g, h \in \mathcal{T}$, we have that $X_{g||h}$ is a closure of $g || h$.

Then we find

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For $X_{r_e || r_f}$, we find another inequation, et cetera. . .

Lemma

Continuing this, we get a finite system of inequations $\langle M, \vec{b} \rangle_{e || f}$.

Theorem

Let $e \otimes f$ be the least solution to $X_{e \parallel f}$ in $\langle M, \vec{b} \rangle_{e \parallel f}$. Then the following hold:

1 $e \otimes f \equiv_{\text{CKA}} e \parallel f$

2 $\llbracket e \otimes f \rrbracket_{\text{BKA}} = \llbracket e \parallel f \rrbracket_{\text{CKA}}$

In other words, $e \otimes f$ is a closure of $e \parallel f$.

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If $e \in \mathcal{T}$, then we can compute a term $e_{\downarrow}$ that is a closure of e .

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Theorem

If $e \in \mathcal{T}$, then we can compute a term $e_{\downarrow}$ that is a closure of e .

Corollary

Let $e, f \in \mathcal{T}$ be such that $\llbracket e \rrbracket_{\text{CKA}} = \llbracket f \rrbracket_{\text{CKA}}$; then $e \equiv_{\text{CKA}} f$.

Conclusion

- Axiomatised equality of *closed, rational pomset languages*.
- Results establishes these as the carrier of the free CKA.
- Extends half of earlier Kleene theorem: terms to pomset automata.
- We also obtain a novel (but inefficient) decision procedure.

Further work

- Explore coalgebraic perspective:
 - Efficient equivalence checking through bisimulation?
 - Can completeness be shown coalgebraically?
- Add “parallel star” operator — closure method does not apply.
- Endgame: lift results to KAT, then NetKAT.

Thank you for your attention



Implementation: <https://doi.org/10.5281/zenodo.926651>.

Draft paper: <https://arxiv.org/abs/1710.02787>.